Global-in-time behavior of weak solutions to reaction-diffusion systems with inhomogeneous Dirichlet boundary condition

Michel Pierre, Takashi Suzuki, Haruki Umakoshi

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Abstract

We study reaction diffusion systems describing, in particular, the evolution of concentrations in general reversible chemical reactions. We concentrate on inhomogeneous Dirichlet boundary conditions. We first prove global existence of (very) weak solutions. Then, we prove that these - although rather weak- solutions converge exponentially in L^1 norm toward the homogeneous equilibrium. These results are proven by means of L^2-duality arguments and through estimates provided by the nonincreasing entropy.

Keywords. reaction diffusion systems, Dirichlet conditions, global existence, asymptotic behavior, entropy, convergence to equilibrium.
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1 Introduction

The purpose of the present paper is to study global existence and asymptotic behavior for reaction diffusion systems *with inhomogeneous Dirichlet boundary conditions* which include as a particular case the classical systems modeling reversible reaction processes for a set of chemical species A_i, $1 \leq i \leq n$:

$$\alpha_1 A_1 + \cdots + \alpha_n A_n \rightleftharpoons \beta_1 A_1 + \cdots + \beta_n A_n, \quad \alpha_i, \beta_i \in \mathbb{N} \cup \{0\}. \tag{1}$$

According to the Mass Action law for the reactions and to Fick's law for the diffusion, the concentrations at position x and time t of A_i, denoted by $u_i =$

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$u_i(x,t)$, satisfy the following evolution system

$$u_{it} - d_i \Delta u_i = (\beta_i - \alpha_i) \left(\prod_{j=1}^{n} u_j^{\alpha_j} - \prod_{j=1}^{n} u_j^{\beta_j} \right), \ \ 1 \le i \le n.$$

We will consider more general systems, always *with inhomogeneous Dirichlet boundary conditions*, that is

$$\begin{cases} u_{it} - d_i \Delta u_i = f_i(u) & \text{in } Q_\infty = \Omega \times (0,\infty), \ \ 1 \le i \le n, \\ u_i(x,t) = g_i(x,t) & \text{on } \Gamma_\infty = \partial\Omega \times (0,\infty), \\ u_i(x,0) = u_{i0}(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded connected open subset with smooth boundary $\partial\Omega$ and $d_i \in (0,\infty), 1 \le i \le n$. The data $u_0 = (u_{i0})_{1 \le i \le n}, g = (g_i)_{1 \le i \le n}$ are assumed to be nonnegative. We will throughout assume that $u_0 \in L^\infty(\Omega)$ and, for simplicity, that g is smooth, for instance such that there exist $G_i, i = 1, ..., n$ with

$$\begin{cases} G_i \in C^1\big([0,\infty); C(\overline{\Omega})\big) \cap C\big([0,\infty); C^2(\overline{\Omega})\big), \\ G_i = g_i \ge 0 \ on \ \Gamma_\infty, \ \partial_t G_i - d_i \Delta G_i = 0 \ in \ Q_\infty, \ G_i(\cdot,0) = g_i(\cdot,0). \end{cases} \tag{3}$$

In the system modeling (1) above, the functions f_i are precisely given by

$$f_i(u) = (\beta_i - \alpha_i) \left(\prod_{j=1}^{n} u_j^{\alpha_j} - \prod_{j=1}^{n} u_j^{\beta_j} \right), \quad \forall \, u = (u_i) \in [0,\infty)^n. \tag{4}$$

Although α_i, β_i are integers in the application to the chemical reaction (1), we will more generally assume that

$$\alpha_i, \beta_i \in [1,\infty) \cup \{0\}.$$

We will consider more general nonlinearities f_i. Throughout the paper, they will satisfy

$$f_i : \mathbb{R}^n \to \mathbb{R} \text{ is locally Lipschitz continuous for } 1 \le i \le n. \tag{5}$$

Under this assumption, System (2) has a unique classical solution u local-in-time. We will also throughout assume that the nonlinearity $f = (f_i)_{1 \le i \le n}$ is quasi-positive, which means

$$f_i(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n) \ge 0, \quad \forall \, 1 \le i \le n, \ \forall \, u \in [0,\infty)^n. \tag{6}$$

In this case, the solution u of System (2) is always nonnegative as far as it exists. Obviously (6) is satisfied by the particular f in (4).

As for chemical systems of type (1), we will often assume that there exist $c_i > 0, 1 \le i \le n$, such that

$$\sum_{i=1}^{n} c_i f_i(u) = 0, \quad \text{for } u \in [0,\infty)^n. \tag{7}$$

Existence of the c_i in (1) is nothing but preservation of mass. It actually holds for f_i as in (4) as soon as there exists $i_1, i_2 \in \{1, ..., n\}$ such that $\alpha_{i_1} - \beta_{i_1} > 0$ and $\alpha_{i_2} - \beta_{i_2} < 0$. Then after summing the equations in (2), equality (7) implies

$$\partial_t(c \cdot u) - \Delta(d_c \cdot u) = 0$$

for $c = (c_i)_{1 \leq i \leq n} > 0$, $d_c = (c_i d_i)_{1 \leq i \leq n} > 0$. This guarantees several a priori estimates of the solution via duality arguments at least in the case of homogeneous Neumann boundary conditions ([4, 8, 19, 20]). Some of them may be extended to Dirichlet boundary conditions but not all. Actually some main estimates are missing for nonhomogeneous boundary conditions.

We will consider the general system (2) with $f = (f_i)_{1 \leq i \leq n}$ satisfying (5), (6), (7) or even more generally the following (8) instead of (7):

$$\sum_{i=1}^{n} c_i f_i(u) \leq 0, \quad \text{for all } u \in [0, \infty)^n. \tag{8}$$

We will sometimes also assume that

$$|f(u)| \leq C(1 + |u|^\gamma), \;\; \gamma \in (1, \infty). \tag{9}$$

as it is the case in example (4).

The goal of this paper is to provide several global existence results for System (2) and to prove exponential asymptotic stability of these global solutions when f is as in (4) and $g_i = s_i$ with $\Pi s_i^{\alpha_i} = \Pi s_i^{\beta_i}$.

As expected in these systems, we will deal with different definitions of solutions, and in particular:
1) "Classical solutions" when the $f_i(u) \in L^\infty(Q_T)$ for all $T \in (0, \infty)$ in which case the solutions have classical derivatives and the equation is to be understood in a classical sense.
2) "Weak solutions" as defined next.
3) "Very weak solutions" as used in Theorem 2.

In this paper, first, we show the existence of weak global-in-time solutions for the system (2) when the diffusion rates $d_1, d_2, ..., d_n$ are "quasi-uniform" in the sense of (13) below (see Theorem 1). These solutions may be even classical if the diffusion rates are even closer (see Remark 1). Next we prove in Theorem 2 the convergence of approximate solutions *no matter the values of the d_i*, this for a very general system with dissipating entropy and including (2) with f_i as in (4). The limit is some kind of "very weak solution" for which some properties of "renormalized solution" could be proved (see Remark 2). We prove in Theorem 3 that all these "solutions" are asymptotically exponentially stable for the specific system (2), (4) when the data are compatible with stationary solutions in the sense of (19).

Definition 1 (weak solution) *We say that $u = (u_1, ..., u_n)$ is a weak solution to (2) if the following conditions are satisfied for all $T \in (0, \infty)$ where $Q_T = \Omega \times (0, T)$ and $\Gamma_T = \partial\Omega \times (0, T)$:*

(i) $u_i \in C([0,\infty); L^1(\Omega))$, $\;f_i(u) \in L^1(Q_T)$,

(ii) *For any* $\varphi : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *with continuous* φ, $\partial_t\varphi$, $\nabla_x\varphi$, $\nabla_x^2\varphi$ *and* $\varphi = 0$ *on* $\Gamma_T \bigcup (\Omega \times \{T\})$ *it holds that*

$$\iint_{Q_T} -u_i\varphi_t - d_i u_i \Delta\varphi \; dxdt = \int_\Omega u_{i0}(x)\varphi(x,0)\; dx$$
$$+ \iint_{Q_T} f_i(u)\varphi \; dxdt - \iint_{\Gamma_T} g_i \partial_\nu \varphi \; dSdt, \quad 1 \le i \le n.$$

To state our result, let us introduce

$$a = \min_i d_i, \quad b = \max_i d_i, \text{ where } 0 < a \le b < +\infty.$$

Let furthermore $C_{m,q} \in (0,\infty)$ be the best constant in the estimate

$$\|\Delta v\|_{L^q(Q_T)} \le C_{m,q}\|F\|_{L^q(Q_T)} \tag{10}$$

where $v : \overline{Q_T} \to \mathbb{R}$ is the solution of the backward heat equation with homogeneous Dirichlet boundary condition:

$$-(v_t + m\Delta v) = F \ge 0 \text{ in } Q_T, \quad v = 0 \text{ on } \Gamma_T, \quad v(x,T) = 0 \text{ in } \Omega. \tag{11}$$

For instance by Corollary 7.31 in [16] or Theorem 6.2 in [25], inequality (10) is valid for each $q \in (1,\infty)$ (see also Lemma 2.1 in [4]).

As a standard approximation of Problem (2), we will consider the solution $u^k = (u_1^k, ..., u_n^k)$ of

$$\begin{cases} \text{for } 1 \le i \le n, \\ u_{it}^k - d_i \Delta u_i^k = \frac{f_i(u^k)}{1 + k^{-1}\sum_{j=1}^n |f_j(u^k)|} \quad \text{in } Q_\infty, \\ u_i^k = g_i \text{ on } \Gamma_\infty, \quad u_i^k(\cdot,0) = u_{i0} \ge 0 \text{ in } \Omega. \end{cases} \tag{12}$$

Since the nonlinearity is uniformly bounded (by k), there exists a global-in-time classical and nonnegative solution $u^k = (u_i^k)_{1\le i\le n} \ge 0$, $1 \le i \le n$, for each k.

Theorem 1 *Assume (5), (6), (8), (9). If moreover*

$$\frac{b-a}{2} C_{\frac{a+b}{2},\gamma'} < 1, \tag{13}$$

then, a subsequence of the solutions $(u^k)_{k\ge 0}$ *of (12) converges in* $L^\gamma(Q_T)^n$ *and* $C([0,T]; L^1(\Omega)^n)$ *for all* $T > 0$. *Moreover, any limit of such converging subsequences is a weak solution of System (2). If* $\gamma = 2$, *then* (13) *is satisfied for all* $0 < a \le b < +\infty$.

Remark 1 We may even obtain classical solutions in Theorem 12 if $b - a$ is smaller than in (13). This is the case if

$$\frac{(b-a)}{2} C_{\frac{a+b}{2},q'} < 1 \text{ where } q' < \frac{\gamma}{\gamma - 2/(N+2)}. \tag{14}$$

Indeed, in this case we obtain (see Remark 6 after the proof of Theorem 1) that u^k is bounded in $L^q(Q_T)$ where $q > (N+2)\gamma/2$. Going back to the equation (2) and using (9), we deduce that u^k is bounded in $L^\infty(Q_T)$ and the solution at the limit is classical.

The result of Theorem 1 does not provide global existence for the system modeling the chemical reaction (1) when the α_i, β_i are quite larger than 2 and when the d_i are not close enough to each other. Actually, this is known as a rather difficult and open question. It was significantly analyzed in the case of Neumann boundary conditions in [12]: there the solutions of the approximate System (12) are proved to converge a.e. up to a subsequence and the limit is a *renormalized solution* in the spirit of [9], but with an adequate definition for this kind of systems as introduced in [12].

Here, we are able to prove a similar convergence result in the case of nonhomogeneous Dirichlet boundary conditions, no matter the values of the d_i. The situation is not so easy since it does not lead to a priori estimates as good as with Neumann boundary conditions, but they nevertheless provide good enough compactness properties for the approximate solutions, at least locaaly inside Ω. As in [12], they strongly rely on the entropy inequality valid for System (2) with f as in (4), namely

$$\sum_{i=1}^{n} (\log u_i) f_i(u) \leq 0. \tag{15}$$

Theorem 2 *Assume (5), (6), (15). Then a subsequence of the solution $(u^k)_{k\geq 0}$ of (12) converges in $L^2(Q_T)^m$ for all $T > 0$.*

Remark 2 If the d_i are close enough so that (13) (resp. (14)) is satisfied, then the limit obtained in Theorem 2 is a weak (resp. a classical) solution of (2). For general d_i's, using truncations as in (39) and the functions $T_r(u_i^k + \eta \sum_{j\neq i} u_j^k)$, we could prove that the limit is a *renormalized solution inside* Q_T in the following sense inspired from [12]. We denote

$$\Psi := \{\psi \in C^2(\mathbb{R}^n; (0,\infty))^+ \text{ with } \partial_i\psi \text{ compactly supported for } 1 \leq i \leq n\}$$

where $\partial_i \psi(u)$ is a notation for the derivative of $u_i \in \mathbb{R} \to \psi(u_1, ..., u_i, ..., u_m)$.

Starting formally from $\partial_t u_i - d_i \Delta u_i = f_i(u)$, we have for all $\psi \in \Psi$

$$\partial_t \psi(u) = \sum_i \partial_i \psi(u) \partial_t u_i = \sum_i \partial_i \psi(u)[d_i \Delta u_i + f_i(u)]. \tag{16}$$

And this may be rewritten

$$\partial_t \psi(u) = \sum_i \left\{ d_i [\nabla \cdot (\partial_i \psi(u) \nabla u_i) - \sum_j \partial_j \partial_i \psi(u) \nabla u_j \nabla u_i] + \partial_i \psi(u) f_i(u) \right\}. \tag{17}$$

This equation may be understood in the sense of distributions in Q_T as soon as

$$\chi_{[u_i \leq r]} \nabla u_i \in L^2_{loc}(Q_T) \text{ for all } r \in (0,\infty),\ T > 0,\ 1 \leq i \leq n. \tag{18}$$

Indeed, since $\partial_i\psi$ is compactly supported for $1 \le i \le n$, we then have

$$\partial_i\psi(u)f_i(u) \in L^\infty(Q_T),\;\; \partial_i\psi(u)\nabla u_i \in L^2_{loc}(Q_T),\; \partial_j\partial_i\psi(u)\nabla u_j\nabla u_i \in L^1_{loc}(Q_T).$$

And as proved later in (34), the estimate (18) will indeed hold here. Note that the estimate is local inside Ω and it is not clear how to extend it up to the boundary except in some cases (see Remark 7).

Since our goal here is to mainly concentrate on the asymptotic behavior of the solutions, and since we do not need to know (17) for doing so, we will not prove it here. Actually, it is an interesting point to see that we can control the asymptotic behavior of the "very weak solutions" without knowing much about them.

Thus a main result of this paper is the exponential stability of the limit "solutions" of (2)-(4) in the case when

$$g_j(x,t) \equiv s_j > 0, \quad \prod_{j=1}^n s_j^{\alpha_j} = \prod_{j=1}^n s_j^{\beta_j}. \tag{19}$$

Then $u_i = s_i > 0$, $1 \le i \le n$, is a spatially homogeneous stationary state of (2)-(4).

Notation. $\|\cdot\|_p$, $1 \le p \le \infty$ will denote the standard L^p norm on Ω.

Theorem 3 *Assume f is given by (4) with (19). Then, the approximate solutions $(u^k)_{k\ge 0}$ of (12) lie in a compact set of $L^2(Q_T)$ for all $T > 0$. There exist positive constants C_1, C_2 such that, for any limit u of converging subsequences*

$$\|u_i(\cdot,t) - s_i\|_1 < C_1 \exp(-C_2 t), \quad \textit{for all}\, t \ge 0, 1 \le i \le n. \tag{20}$$

Several existence results of global-in-time solutions and their asymptotic behavior have been known for the reaction diffusion system associated with (1), particularly, when the boundary condition is of homogeneous Neumann type. First, when $n = 3$ with $f_1 = -u_1^{\alpha_1}u_2^{\alpha_2} + u_3^{\beta_3} = f_2 = -f_3$, existence results of global classical solutions are proved in [14] in particular when $\beta_3 > \alpha_1 + \alpha_2$ and for some other particular situations. Exponential convergence towards the stationary solutions is proved in [10] for these f_i for all $\alpha_1, \alpha_2, \beta_3 \ge 1$ (see also [6] for other results with $n = 3$).

When $n = 4$ and $f_i = (-1)^i(u_1u_3 - u_2u_4)$, weak solutions exist globally in time for any space dimension N (see [8]). Furthermore, classical solutions exist globally in time if $N \le 2$ (see [4], [13]) or in any dimension if the diffusion coefficients are quasi-uniform in the sense of (13) (see [4]). Exponential asymptotic stability for the L^1-norm is proved in [5], [7].

For the general system (2),(4), weak (resp. classical) solutions exist globally in time when the diffusion coefficients are quasi-uniform in the sense of (13) (resp. (14)) (see [4]). Finally, global renormalized solutions are proved to exist in [12] for rather general systems with general diffusions and Neumann type of boundary conditions. And their asymptotic behavior is analyzed in [21].

Inhomogeneous Dirichlet boundary condition are studied in [11]. They are concerned with the case $n = 3$, $\alpha_1 = \beta_2 = \alpha_3 = 1$, and $\beta_1 = \alpha_2 = \beta_3 = 0$:

$$\begin{cases} u_t - d_1\Delta u = -u + vw & \text{in } \Omega \times (0,T) \\ v_t - d_2\Delta v = u - vw & \text{in } \Omega \times (0,T) \\ w_t - d_3\Delta w = -u + vw & \text{in } \Omega \times (0,T) \\ u(x,t) = a,\ v(x,t) = b,\ w(x,t) = c & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0,\ w(x,0) = w_0(x) > 0 \text{ in } \Omega. \end{cases} \tag{21}$$

If a, b, c are positive constants satisfying $a = bc$, there exists a classical solution $(u, v, w) = (u(\cdot,t), v(\cdot,t), w(\cdot,t))$ global-in-time and it holds that

$$\lim_{t\to\infty} (u(\cdot,t), v(\cdot,t), w(\cdot,t)) = (a, b, c) \quad \text{in } C^{\nu}(\overline{\Omega})$$

where $1 < \nu < 2$. However, asymptotic behavior of the global-in-time "weak" solution has not been studied for the general case of (2)-(4).

2 Proof of Theorem 1

We first show an estimate on the solution of a parabolic differential inequality. It is similar to Proposition 1.1 of [4], but with nonhomogeneous Dirichlet boundary conditions.

Given $\alpha \in (0,1)$, we take $M = M(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0,T])$ satisfying

$$0 < a \le M(t,x) \le b < \infty, \quad (x,t) \in Q_T. \tag{22}$$

We consider the parabolic differential inequality

$$\begin{cases} u_t - \Delta(Mu) \le 0 \text{ in } Q_T, \\ u = g \text{ on } \Gamma_T, \quad u(\cdot,0) = u_0(x) \ge 0 \text{ in } \Omega. \end{cases} \tag{23}$$

We will estimate $\|u\|_{L^p(Q_T)}$ for $p \in [2,\infty)$, under the assumption

$$C_{\frac{a+b}{2},p'} \cdot \frac{b-a}{2} < 1, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{24}$$

where $C_{m,q} \in (0,\infty)$ stands for the best constant in the parabolic regularity (10)-(11).

Remark 3 We have $C_{\frac{a+b}{2},2} \le \frac{2}{a+b}$ so that (24) is always satisfied for $p' = 2$ and for all $0 < a < b < \infty$. Indeed, multiplying (11) by $-\Delta v$ leads to

$$\iint_{Q_T} v_t\Delta v + m(\Delta v)^2 = \iint_{Q_T} -F\Delta v \le \|F\|_{L^2(Q_T)}\|\Delta v\|_{L^2(Q_T)}.$$

We then use $\iint_{Q_T} v_t\Delta v = \frac{1}{2}\int_\Omega |\nabla v(x,0)|^2 \ge 0$ to deduce $C_{m,2} \le 1/m$.

Remark 4 It is interesting to notice that the condition (24) is "open" with respect to p' in the sense that if (24) holds with p', then it holds with $(p+\epsilon)'$ for ϵ small enough. Indeed, the $C_{m,q}$ has the property: $C^-_{m,q} := \liminf_{\eta\to 0^+} C_{m,q-\eta} \le C_{m,q}$. To see it, let q_η satisfy

$$\frac{1}{q_\eta} = \frac{1}{2}\left[\frac{1}{q} + \frac{1}{q-\eta}\right] \text{ i.e. } q_\eta = q - \eta q/(2q-\eta).$$

By the Riesz-Thorin interpolation theorem (see e.g. [18], chapter 2) applied to the mapping $F \mapsto \Delta v$ in (11), we have

$$C_{m,q_\eta} \le C^{1/2}_{m,q} C^{1/2}_{m,q-\eta} \Rightarrow C^-_{m,q} \le C^{1/2}_{m,q}(C^-_{m,q})^{1/2} \Rightarrow C^-_{m,q} \le C_{m,q}.$$

Notation. For the boundary Γ_T, we will use dS, ν, ∂_ν to denote respectively the surface element, the exterior unit normal and the exterior normal derivative.

For the solution of equation (11), we can define the best constant $E_{m,q,T} \in (0,\infty)$ for the inequality

$$\iint_{\Gamma_T} |\partial_\nu v|\; dSdt \le E_{m,q,T}\|F\|_{L^q(Q_T)}, \quad q > 1, \tag{25}$$

using the trace embedding $W^1_q(\Omega) \hookrightarrow L^1(\partial\Omega)$.

Proposition 4 *Let $u \ge 0$ be a classical solution to (23) with $M \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0,T])$ satisfying (22) and (24). Then it holds that*

$$\|u\|_{L^p(Q_T)} \le (1 + bD_{a,b,p'})\, T^{\frac{1}{p}}\|u_0\|_p + \tilde{E}_{\frac{a+b}{2},p',T} \cdot b \cdot \|g\|_{L^\infty(\Gamma_T)} \tag{26}$$

for $p \in [2,\infty)$, where

$$D_{a,b,p'} = \frac{C_{\frac{a+b}{2},p'}}{1 - C_{\frac{a+b}{2},p'} \cdot \frac{b-a}{2}}, \quad \tilde{E}_{a,b,p',T} = E_{\frac{a+b}{2},p',T}\left(1 + \frac{b-a}{2}D_{a,b,p'}\right).$$

Remark 5 Note that, according to Remark 3, $D_{a,b,2} < +\infty$ so that any u satisfying (23) is bounded in $L^2(Q_T)$ for all $T > 0$ with a bound depending on $\|u_0\|_{L^2(\Omega)^n}, \|g\|_{L^\infty(\Gamma_T)^n}$.

To prove Proposition 4, we begin with a parabolic estimate for the dual problem

$$\psi_t + M\Delta\psi = -\Theta \text{ in } Q_T, \quad \psi(T,x) = 0 \text{ in } \Omega, \quad \psi(x,t) = 0 \text{ on } \Gamma_T, \tag{27}$$

where $\Theta \in C^\infty_0(Q_T)$. This inequality will be proved similarly as in Lemma 2.2 of [4] concerning homogeneous Neumann boundary condition.

Lemma 5 *For $M = M(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0,T])$ satisfying (22) and (24) and $1 < p' \le 2$, the following holds for the solution ψ of (27)*

$$\left\{\begin{array}{l} \|\Delta\psi\|_{L^{p'}(Q_T)} \le D_{a,b,p'}\|\Theta\|_{L^{p'}(Q_T)} \\ \|\psi(\cdot,0)\|_{p'} \le (1 + bD_{a,b,p'})T^{1/p}\|\Theta\|_{L^{p'}(Q_T)} \end{array}\right. \tag{28}$$

Proof: By the standard theory (e.g. Corollary 7.31 and Theorem 7.32 in [16]) or Theorem 6.2 in [25]), Problem (27) admits a unique classical solution $\psi = \psi(x,t)$. We write (27) as

$$\psi_t + \frac{a+b}{2}\Delta\psi = \left(\frac{a+b}{2} - M\right)\Delta\psi - \Theta. \tag{29}$$

Then (10) implies

$$\begin{aligned}\|\Delta\psi\|_{L^{p'}(Q_T)} &\leq C_{\frac{a+b}{2},p'}\left\|\left(\frac{a+b}{2} - M\right)\Delta\psi - \Theta\right\|_{L^{p'}(Q_T)} \\ &\leq C_{\frac{a+b}{2},p'}\left\{\left(\frac{b-a}{2}\right)\|\Delta\psi\|_{L^{p'}(Q_T)} + \|\Theta\|_{L^{p'}(Q_T)}\right\}\end{aligned}$$

where we used: $\|\frac{a+b}{2} - M\|_{L^\infty(Q_T)} \leq \frac{b-a}{2}$. Therefore,

$$\left\{1 - C_{\frac{a+b}{2},p'}\frac{b-a}{2}\right\}\|\Delta\psi\|_{L^{p'}(Q_T)} \leq C_{\frac{a+b}{2},p'}\|\Theta\|_{L^{p'}(Q_T)}.$$

This is the first inequality of (28) (we use (24) here). The second inequality is derived from $-\psi(0) = \int_0^T \psi_t(\cdot,t)dt$ and (27) which imply

$$\|\psi(0)\|_{p'} \leq T^{1/p}\|\psi_t\|_{L^{p'}(Q_T)} \leq T^{1/p}[b\,\|\Delta\psi\|_{L^{p'}(Q_T)} + \|\Theta\|_{L^{p'}(Q_T)}].$$

$\square$

Proof of Proposition 4: If $0 \leq \Theta \in C_0^\infty(Q_T)$, the classical solution to (27) satisfies $\psi = \psi(x,t) \geq 0$. Then both u and ψ are nonnegative and we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\psi\ dx &= \int_\Omega u_t\psi + u\psi_t\ dx \\ &\leq \int_\Omega [\Delta(Mu)]\psi + u(-M\Delta\psi - \Theta)\ dx.\end{aligned}$$

Since

$$\int_\Omega [\Delta(Mu)]\psi - (Mu)\Delta\psi\ dx = -\int_{\partial\Omega} gM\partial_\nu\psi\ dS,$$

it holds that

$$\frac{d}{dt}\int_\Omega u\psi\ dx \leq -\int_\Omega u\Theta\ dx - \int_{\partial\Omega} gM\partial_\nu\psi\ dS. \tag{30}$$

Here we use (25), (29), and (28) to conclude

$$\iint_{\Gamma_T} |\partial_\nu\psi|\ dSdt \leq E_{\frac{a+b}{2},p',T}\left(1 + \frac{b-a}{2}D_{a,b,p'}\right)\|\Theta\|_{L^{p'}(\Omega_T)}. \tag{31}$$

Inequalities (30) and (31) imply

$$\iint_{Q_T} u\Theta \; dxdt \leq \|u_0\|_p \|\psi(\cdot,0)\|_{p'} + \|g\|_{L^\infty(\Gamma_T)} \cdot b \iint_{\Gamma_T} |\partial_\nu \psi| \; dSdt$$
$$\leq \left\{ (1 + bD_{a,b,p'}) T^{\frac{1}{p}} \|u_0\|_p + \tilde{E}_{a,b,p',T} b \|g\|_{L^\infty(\Gamma_T)} \right\} \|\Theta\|_{L^{p'}(Q_T)}. \tag{32}$$

Inequality (32), valid to any $0 \leq \Theta \in C_0^\infty(Q_T)$, implies (26) by duality since $u \geq 0$. □

Proof of Theorem 1: Let us consider the global regular solution u^k of the approximate problem (12). Recalling (8), let

$$v^k = c \cdot u^k, \quad h = c \cdot g, \quad v_0 = c \cdot u_0,$$

where $c = (c_i)$, $u^k = (u_i^k)$, $g = (g_i)$ and $u_0 = (u_{i0})$. Let also $d_c := (d_i c_i)$. Then (8) implies

$$v_t^k - \Delta(M v^k) \leq 0 \text{ in } Q_T$$
$$v^k(x,t) = h(x,t) \text{ on } \Gamma_T, \quad v^k(x,0) = v_0(x) \geq 0 \text{ in } \Omega$$

with $M = M(x,t) = d_c \cdot u^k / c \cdot u^k$ which satisfies (22) with $a = \min_i d_i, b = \max_i d_i$.

According to Remark 4, the assumption (13) implies that $\frac{b-a}{2} C_{\frac{a+b}{2},(\gamma+\epsilon)'} < 1$ for some $\epsilon > 0$. By Proposition 4, $\|v^k\|_{L^{\gamma+\epsilon}(Q_T)} \leq C_T$ for all $T > 0$. It follows by (9) that $f_i(u^k)$ is bounded in $L^{1+\eta}(Q_T)$ for $\eta = \epsilon/\gamma > 0$. We then may use the L^1-compactness property of the heat operator saying (see e.g. [2], [1]) that the mapping $(w_0, F) \in L^1(\Omega) \times L^1(Q_T) \to w \in L^1(Q_T)$ is compact where w is the solution of

$$w_t - m\Delta w = F \; in \; Q_T, \; w = 0 \; on \; \Gamma_T, \; w(\cdot,0) = w_0. \tag{33}$$

Applying this here to

$$m = d_i, \; F = f_i(u^k)/(1 + k^{-1} \sum_j |f_j(u^k)|), \; w = u_i^k - G_i, w_0 = u_{i0}^k - g_i(0),$$

where G_i is defined in (3), we deduce that u^k lies in a compact set of $L^1(Q_T)^m$. Up to a subsequence, we may assume that, for all $T > 0$, u^k converges in $L^1(Q_T)^m$ and a.e. to some u which, by Fatou's Lemma, belongs to $L^1(Q_T)^m$. It implies that $f_i(u^k)$ converges a.e. to $f_i(u)$ for all i. Since $f_i(u^k)/(1 + k^{-1} \sum_j |f_j(u^k)|)$ is bounded in $L^{1+\eta}(Q_T)$, we deduce by Egorov's theorem that the convergence holds also in $L^1(Q_T)$. Now we may pass to the limit in

$$\iint_{Q_T} -u_i^k \varphi_t - d_i u_i^k \Delta\varphi \; dxdt = \int_\Omega u_{i0}^k \varphi \; dx$$
$$+ \iint_{Q_T} \frac{f_i(u^k)}{1 + k^{-1} \sum_j |f_j(u^k)|} \varphi \; dxdt - \iint_{\Gamma_T} g_i \partial_\nu \varphi \; dSdt, \quad 1 \leq i \leq n,$$

for all φ as in Definition 1. To conclude that u is a weak solution, we only need to check that $u \in C([0,\infty); L^1(\Omega)^n)$. This follows from the L^1-contraction property of the heat operator, namely

$$\|u_i^k(t) - u_i^p(t)\|_1 \le \|u_{i0}^k - u_{i0}^p\|_1 + \int_0^t \|\frac{f_i(u^k)}{1+k^{-1}\sum_j |f_j(u^k)|} - \frac{f_i(u^p)}{1+p^{-1}\sum_j |f_j(u^p)|}\|_1 dt.$$

This proves that u^k converges in $L^\infty([0,T] : L^1(\Omega)^n)$ and the limit is therefore continuous from $[0,\infty)$ into $L^1(\Omega)$. □

Remark 6 If we replace (13) by the (stronger) assumption (14), then by Proposition 4, and the same proof as above, u^k is bounded in $L^q(Q_T)$ for $q > (N+2)\gamma/2$. This implies by (9) that $f_i(u^k)$ is bounded in $L^s(Q_T)$ for some $s > (N+2)/N$. And it is well-known (see e.g. [15]) that u_i^k is then bounded in $L^\infty(Q_T)$ and so is $f_i(u^k)$. The limit of u^k is then a classical solution of (2).

3 Proof of Theorem 2

Let us first prove the following estimate for the solution u^k of (12). For $\delta > 0$, we denote $\Omega_\delta = \{x \in \Omega; d(x, \partial\Omega > \delta\}$. Then

$$\iint_{[u_i^k \le r]\cap\Omega_\delta} |\nabla u_i^k|^2 \le C_\delta\, r \text{ for all } r \in [0,\infty), 1 \le i \le n, k \in \mathbb{N}. \tag{34}$$

In the following computation, for simplicity, we drop the ^{k} in the notation. Let us introduce $w_i := u_i \log u_i + 1 - u_i \ge 0$. We have

$$\partial_t \sum_i w_i - \Delta \sum_i d_i w_i = \sum_i \log u_i f_i(u) - \sum_i 4d_i|\nabla\sqrt{u_i}|^2 \le -\sum_i 4d_i|\nabla\sqrt{u_i}|^2, \tag{35}$$

the last inequality coming from the assumption (15). Let φ be the first eigenfunction of the Dirichlet-Laplacian on the open connected set Ω, namely

$$-\Delta\varphi = \lambda_1\varphi \; in \; \Omega, \;\; \varphi = 0 \; on \; \partial\Omega, \; \|\varphi\|_\infty = 1, \; \varphi > 0 \; on \; \Omega. \tag{36}$$

Multiplying the previous inequality by φ and integrating on Q_T gives

$$\begin{cases} \int_\Omega \varphi \sum_i w_i(T) + \iint_{Q_T} \varphi \sum_i d_i(\lambda_1 w_i + 4|\nabla\sqrt{u_i}|^2) \\ \le \int_\Omega \varphi \sum_i w_i(0) - \iint_{\Gamma_T} \partial_\nu\varphi \sum_i d_i(g_i \log g_i + 1 - g_i). \end{cases}$$

We deduce that for some $C = C(\max_i\{\|g_i\|_\infty, \|u_{i0}\log u_{i0}\|_1\}, \|\partial_\nu\varphi\|_1) < \infty$

$$\max_i \int_{Q_T} 4\varphi|\nabla\sqrt{u_i}|^2 = \max_i \int_{Q_T} \varphi\frac{|\nabla u_i|^2}{u_i} \le C, \tag{37}$$

and the estimate (34) follows with $C_\delta = C/\min_{x\in\Omega_\delta}\varphi(x)$. □

The same inequality (35) implies also the following L^2-estimate:

$$\max_i \|u_i^k \log u_i^k\|_{L^2(Q_T)} \le C = C(\max_i\{\|g_i\|_\infty, \|u_{i0}\log(u_{i0})\|_2, T, a, b) < +\infty. \tag{38}$$

Proof. Indeed, (35) implies

$$\partial_t(\sum_i w_i) - \Delta(M\sum_i w_i) \le 0, \quad M := \sum_i d_i w_i / \sum_i w_i.$$

Thanks to $w_i \ge 0$, we have $a = \min_i d_i \le M \le b = \max_i d_i$. Thus Proposition 4 applied with $p = 2$ (see Remark 5) implies that $\|\sum_i w_i\|_{L^2(Q_T)} \le C$ where C is as in (38). Now, using again the nonnegativity of the w_i and the fact that $s\log s$ is bounded from a bove for s large by $2[s\log s + 1 - s]$, estimate (38) follows.□

Proof of Theorem 2. Let us prove the convergence of u^k in $L^2(Q_T)$ for all $T > 0$. We will first prove that u^k converges a.e. on Q_∞. Then the $L^2(Q_T)$ convergence will follow from the estimate (38).

For all $r \in (0,\infty)$, we introduce $T_r \in C^2([0,\infty);[0,\infty))$ with

$$\left\{ \begin{array}{l} 0 \le T_r'(s) \le 1 \text{ and } T_r''(s) \le 0 \text{ for all } s \in [0,\infty), \\ T_r(s) = s \text{ for } s \in [0,r],\ T_r'(s) = 0 \ \text{ for } s \in [2r,\infty). \end{array} \right. \tag{39}$$

Let now $v_i^k = u_i^k + \eta U_i^k$ where $U_i^k = \sum_{j\ne i} u_j^k, \eta > 0$. We also denote $F_i = f_i/(1 + k^{-1}\sum_j |f_j|)$. Then

$$\left\{ \begin{array}{l} \partial_t T_r(v_i^k) - d_i \Delta T_r(v_i^k) = \\ T_r'(v_i^k)\left[F_i(u^k) + \eta\sum_{j\ne i}[F_j(u^k) + (d_j - d_i)\Delta u_j^k]\right] - d_i T_r''(v_i^k)|\nabla v_i^k|^2. \end{array} \right. \tag{40}$$

Let us analyze each of the terms in the right-hand side of this equality. We will repeatedly use that

$$[v_i^k \le 2r] \quad \Rightarrow \quad [u_i^k \le 2r] \text{ and } [u_j^k \le 2r/\eta]\ \forall j \ne i.$$

In the following, r is fixed arbitrarily in $(0,\infty)$ and $\eta > 0$ is fixed small enough (to be made precise later). By the choice of T_r and of v^k

$$T_r'(v_i^k)[F_i(u^k) + \eta\sum_{j\ne i} F_j(u^k)] \text{ is bounded in } L^\infty(Q_T) \text{ independently of } k. \tag{41}$$

We also have, with $Q_T^\delta := (0,T)\times\Omega_\delta$, (recall that T_r'' vanishes outside $[0,2r]$)

$$\|T_r''(v_i^k)^{1/2}\nabla v_i^k\|_{L^2(Q_T^\delta)} \le C\left\{\|\chi_{[u_i^k\le 2r]}\nabla u_i^k\|_{L^2(Q_T^\delta)} + \sum_{j\ne i}\|\chi_{[u_j^k\le 2r/\eta]}\nabla u_j^k\|_{L^2(Q_T^\delta)}\right\}.$$

Thus, by (34)

$$T_r''(v_i^k)|\nabla v_i^k|^2 \text{is bounded in } L^1(Q_T^\delta)\ \forall\,\delta > 0. \tag{42}$$

Now we write

$$T_r'(v_i^k)\Delta u_j^k = \nabla\cdot(T_r'(v_i^k)\nabla u_j^k) - T_r''(v_i^k)\nabla v_i^k\nabla u_j^k,$$

$$\|T'(v_i^k)\nabla u_j^k\|_{L^2(Q_T^\delta)} \le C\|\chi_{[u_j^k\le 2r/\eta]}\nabla u_j^k\|_{L^2(Q_T^\delta)},$$

$$\|T_r''(v_i^k)\nabla v_i^k\nabla u_j^k\|_{L^1(Q_T^\delta} \le C\|T_r''(v_i^k)^{1/2}\nabla v_i^k\|_{L^2(Q_T^\delta)}\|\chi_{[u_j^k\le 2r/\eta]}\nabla u_j^k\|_{L^2(Q_T^\delta)}.$$

We deduce, by using (34) again, that

$$T_r'(v_i^k)\Delta u_j^k \text{ is bounded in } L^2(0,T;H^{-1}(\Omega_\delta)) + L^1(0,T;L^1(\Omega_\delta))\ \forall\,\delta>0. \quad (43)$$

Let $\psi\in C_0^\infty(\Omega)^+$. We deduce from (41), (42), (43) and equation (40) that $\partial_t(\psi T_r(v_i^k)) - d_i\Delta(\psi T_r(v_i^k))$ is bounded in $L^1(Q_T)+L^2(0,T;H^{-1}(\Omega))$. Since moreover $\psi T_r(v_i^k)$ is bounded by $r\|\psi\|_\infty$ and vanishes on Γ_T, it follows that $\big(\psi T_r(v_i^k)\big)_{k\ge 0}$ lies in a compact set of $L^1(Q_T)$: to see this, we may use the L^1-compactness as stated for w in (33) and the fact that, if in (33), F is bounded in $L^1(0,T;H^{-1}(\Omega))$ and $w_0=0$ then w lies in a compact set of $L^2(Q_T)$ (see e.g. [17], Théorème 5.1).

Using a diagonal extraction process, we can deduce that there exists a subsequence of v_i^k (still denoted v_i^k) such that $T_r(v_i^k)$ converges a.e. on Q_∞ for all $r\in(0,\infty)$ (here we fix η small enough as indicated below).

This implies that v_i^k converges a.e. on Q_∞ itself. Indeed, let us denote by w_r the pointwise limit of $T_r(v_i^k)$ and let $K_r=[w_r<r]$. For $(x,t)\in K_r$, $T_r(v_i^k(x,t)) = v_i^k(x,t)$ for k large enough so that $v_i^k(x,t)$ converges to $w_r(x,t)$. Therefore v_i^k converges a.e. on K_r. On the other hand, since, thanks to (38), v_i^k is bounded in $L^1(Q_T)$ and we have (using Fatou's lemma for $\Rightarrow$)

$$+\infty > C \ge \int_{Q_T} 2v_i^k \ge \int_{Q_T} T_r(v_i^k) \ \Rightarrow\ C\ge \int_{Q_T} w_r \ge r|[w_r\ge r]\cap Q_T|.$$

Thus $\lim_{r\to\infty}|[w_r\le r]\cap Q_T| = 0$ and $\cup_{r\in(0,\infty)}[w<r]\cap Q_T = Q_T$ a.e.. Whence the a.e. convergence of v_i^k on Q_∞.

Since the $n\times n$ matrix A with 1 on its diagonal and η elsewhere is invertible for η small, and since $v^k = Au^k$, it follows that u_i^k converges also a.e. on Q_∞. We now use the fact that $u_i^k\log u_i^k$ is bounded in $L^2(Q_T)$ by (38). Together with the a.e. convergence this implies the convergence of u_i^k in $L^2(Q_T)$ by Egorov's theorem again. □

4 Proof of Theorem 3

Here, f_i is defined by (4) and $g_i=s_i$ so that the approximate problem (12) becomes

$$u_{it}^k - d_i\Delta u_i^k = (\beta_i-\alpha_i)\frac{R(u^k)}{1+k^{-1}\sum_j|f_j(u^k)|} \quad \text{in } Q_T$$

$$u_i^k = s_i \text{ on } \Gamma_T,\quad u_i^k(x,0)=u_{i0}^k(x) \text{ in } \Omega,\quad 1\le i\le n,$$

$$\text{where } R(u) = \prod_{j=1}^n u_j^{\alpha_j} - \prod_{j=1}^n u_j^{\beta_j}.$$

Here (9) holds with $\gamma = \max\{\sum_i \alpha_i, \sum_i \beta_i\}$. Moreover (15) is satisfied since

$$\sum_{i=1}^{n} f_i(u)\log u_i = R(u)\sum_{i=1}^{n}(\beta_i - \alpha_i)\log u_i = R(u)\{\log\prod_{i=1}^{n} u_i^{\beta_i} - \log\prod_{i=1}^{n} u_i^{\alpha_i}\} \le 0. \quad (44)$$

Therefore we can apply the results of Theorem 2. But, we will have more estimates here due to the choice of the s_i. Let us introduce

$$L(u_i^k, s_i) \equiv u_i^k(\log u_i^k - \log s_i) + s_i - u_i^k \ge 0, \quad (45)$$

the nonnegativity coming from $\xi \ge \log\xi + 1$ for $\xi > 0$ and applied to $\xi = s_i/u_i^k$.

Lemma 6

$$\zeta^k(x,t) = \sum_{i=1}^{n} L(u_i^k(x,t), s_i), \quad \zeta_0(x) = \sum_{i=1}^{n} L(u_{i0}(x), s_i) \quad (46)$$

it holds that

$$\begin{cases} \zeta_t^k - \sum_{i=1}^{n} d_i \Delta L(u_i^k, s_i) + \sum_i d_i \frac{|\nabla u_i^k|^2}{u_i^k} \le 0 \ \textit{in}\ Q_T \\ \zeta^k = 0 = \partial_\nu \zeta^k \ \textit{on}\ \Gamma_T, \quad \zeta^k(\cdot, 0) = \zeta_0 \ge 0 \ \textit{in}\ \Omega. \end{cases} \quad (47)$$

Proof: We have $u_j^k = u_j^k(x,t) > 0$ for $t > 0$, and then it follows that

$$\begin{aligned} &\partial_t L(u_i^k, s_i) - d_i \Delta L(u_i^k, s_i) \\ &= \sum_i \left\{(\log u_i^k - \log s_i)(u_{it}^k - d_i \Delta u_i^k) - d_i \frac{|\nabla u_i^k|^2}{u_i^k}\right\} \\ &= \sum_i \left\{(\log u_i^k - \log s_i)(\beta_i - \alpha_i)\frac{R(u^k)}{1 + k^{-1}\sum_j |f_j(u^k)|} - d_i \frac{|\nabla u_i^k|^2}{u_i^k}\right\}. \end{aligned}$$

We already checked that $\sum_i \log u_i^k(\beta_i - \alpha_i)R(u^k) \le 0$ (see (44)). On the other hand, $\sum_i \log s_i(\beta_i - \alpha_i) = 0$ due to the assumption (19). The first estimate of (47) follows. The boundary conditions follow from $u_i^k = s_i$ at the boundary so that $L(u_i^k, s_i) = 0, \partial_\nu L(u_i^k, s_i) = \partial_\nu u_i^k(\log u_i^k - \log s_i) = 0$. □

Now we use the argument of [24], taking a ball Ω_0 such that $\overline{\Omega} \subset \Omega_0$. Let $\lambda_1 > 0$ and $\varphi = \varphi(x)$, $\|\varphi\|_\infty = 1$ be the first eigenvalue and the associated eigenfunction, respectively:

$$-\Delta\varphi = \lambda_1\varphi, \ \varphi > 0 \ \text{ in } \Omega_0, \quad \varphi = 0 \text{ on } \partial\Omega_0.$$

Let $\delta_\varphi = \inf_\Omega \varphi > 0$.

Lemma 7 *It holds that*

$$\delta_\varphi \int_\Omega \zeta^k(x,t)\ dx \le \int_\Omega \zeta^k(x,0)\ dx \ \cdot e^{-a\lambda_1 t} \quad (48)$$

where $a = \min_i d_i$.

Proof: Thanks to the inequality and the boundary conditions in (47), we have

$$\frac{d}{dt}\int_\Omega \zeta^k\,\varphi\;dx + \int_\Omega \varphi \sum_i d_i \frac{|\nabla u_i^k|^2}{u_i^k} \le \sum_i \int_\Omega \Delta\varphi\, d_i\, L(u_i^k, s_i)dx. \qquad (49)$$

In particular

$$\frac{d}{dt}\int_\Omega \zeta^k\,\varphi\;dx \le -\lambda_1 \sum_{i=1}^n \int_\Omega d_i\, L(u_i^k, s_i)\varphi\;dx \le -\lambda_1 a \int_\Omega \zeta^k\,\varphi\;dx.$$

Whence (48). □

Here we use an elementary inequality.

Lemma 8 (Cziszar-Kullback) *For any measurable functions $f:\Omega\mapsto[0,\infty), g:\Omega\mapsto(0,\infty)$, it holds that*

$$3\left(\int_\Omega |f-g|\;dx\right)^2 \le \left(\int_\Omega (2f+4g)\;dx\right)\left(\int_\Omega [f\log\frac{f}{g} - f + g]\;dx\right). \qquad (50)$$

Proof: Since

$$3|\xi-1|^2 \le (2\xi+4)(\xi\log\xi-\xi+1), \text{for all} \quad \xi>0$$

it follows by choosing $\xi = f(x)/g(x)$ (assuming $f(x)<\infty$) and taking the square root that

$$\sqrt{3}|f(x)-g(x)| \le \big(2f(x)+4g(x)\big)^{1/2}\Big(f(x)\log\frac{f(x)}{g(x)} - f(x) + g(x)\Big)^{1/2}.$$

If the right-hand side of (50) is infinite, then the inequality holds. Therefore, we may assume that it is finite which implies that f and g are finite a.e. We integrate the above pointwise inequality over Ω and apply Schwarz' inequality to the second integral to obtain (50). □

Proof of Theorem 3: By Lemma 7 and Lemma 8 applied with $f = u_i^k$ and $g = s_i$, we obtain

$$\left\{\begin{array}{l} \|u_i^k(\cdot,t)-s_i\|_1^2 \le \frac{1}{3}\cdot\int_\Omega \big(2u_i^k(x,t)+4s_i\big)\;dx\cdot\int_\Omega L(u_i^k,s_i)dx \\ \le \frac{\exp(-a\lambda_1 t)}{3\delta_\varphi}\cdot\int_\Omega \big(2u_i^k(x,t)+4s_i\big)\,dx\cdot\|\zeta_0\|_1. \end{array}\right. \qquad (51)$$

This inequality implies also that (after taking the square root)

$$\|u_i^k\|_1 \le s_i + (\|\zeta_0\|_1/3\delta_\varphi)^{-1/2}\left[2\|u_i^k\|_1 + 4s_i|\Omega|\right]^{1/2}.$$

This implies that

$$\sup_{k\in\mathbb{N},t\ge0}\|u_i^k(\cdot,t)\|_1 < +\infty. \qquad (52)$$

thus we may deduce from (51) that

$$\|u_i^k(\cdot,t) - s_i\|_1^2 \leq C\exp(-a\lambda_1 t)$$

with $C > 0$ independent of k. Then, Theorem 2 and Fatou's Lemma imply (20). $\square$

Remark 7 Going back to the estimate (49), we see that $\sqrt{u_i^k}$ is bounded in $L^2(0,T;H^1(\Omega))$ for all $i = 1,...,n$. In other words, $\sqrt{u_i^k} - \sqrt{g_i}$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Therefore, for any limit u of u^k in Theorem 3, $\left(\sqrt{u_i} - \sqrt{g_i}\right) \in L^2(0,T;H_0^1(\Omega))$. In other words, the nonhomogeneous Dirichlet boundary condition is kept at the limit.

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